CONTRACT DESIGN, ARBITRAGE, AND HEDGING IN THE EURODOLLAR FUTURES MARKET

Don M. Chance*

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*First Union Professor of Financial Risk Management, Department of Finance – 0221, Virginia Tech, Blacksburg, VA 24061; (540) 231-5061; fax: (540) 231-3155; dmc@vt.edu. The author appreciates comments from the faculty seminar of the University of Georgia

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The Eurodollar futures contract at the Chicago Mercantile Exchange is arguably the most successful futures contract of all time. Yet by design the contract is structured such that the futures price cannot converge to the value of the underlying Eurodollar time deposit. Because of this problem, the standard cash and carry arbitrage in which the underlying asset is purchased and the futures sold is not risk-free. Consequently, even in the absence of marking-to-market, the Eurodollar futures rate does not equal the implied forward rate, a statement widely assumed to be true. This paper examines the arbitrage and hedging of Eurodollar futures, demonstrating the characteristics of differences resulting from this unique contract design that arise in arbitrage and in a variety of hedging scenarios. It then provides empirical estimates of the risk introduced into hedges by this aspect of the contract.

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The only perfect hedge is in a Japanese garden.

(Old futures saying)

[*And in academic textbooks and brochures put out by futures exchanges.*]

1. Introduction

One of the most active futures contracts is the Eurodollar futures contract on the Chicago Mercantile Exchange. By many standards, it could easily be declared the most successful futures contract ever. Based on the rate on a 90-day Eurodollar time deposit, a rate called LIBOR for London Interbank Offer Rate, the average daily trading volume in the year 2001 was 730,000 contracts, with each contract covering \$1,000,000 of underlying Eurodollar time deposits. Yet when the Chicago Mercantile Exchange (CME) launched the contract in 1982, it was designed such that one of the most important and common features of all futures contracts is missing: the convergence of the futures price to the price of the underlying spot market instrument. With the exception of differences caused by institutional frictions, such as transaction costs, taxes, delivery options, etc., the spot price converges to the futures price for virtually all futures contracts.[1] This convergence is generally thought to be critical to the pricing of a futures contract. It follows that the standard cash-and-carry argument that leads to the widely-known cost of carry model for pricing futures does not hold for Eurodollar futures and that the futures rate does not equal the implied forward rate, even in the absence of the daily settlement feature.

Instead of structuring the contract to provide convergence of the futures to the underlying, a 90-day Eurodollar time deposit, the CME designed the contract so that its price converges to 100 minus the rate on a 90-day Eurodollar time deposit, with the latter adjusted by the factor 90/360. Hence, the futures contract is priced at expiration as though the underlying is a discount instrument, such as a Treasury bill or commercial paper, but in fact, Eurodollar time deposits are add-on instruments. As a result, the futures price cannot converge to the spot price at expiration.

The problem can be stated in simple algebraic terms: Given the equation $y = 1/(1 + x) - (1 - x)$, with $0 \leq x < 1$, find a solution such that $y = 0$. Figure 1 graphs this function. The only solution is at $x = 0$. What does this translate into for the Eurodollar futures contract? Let x be LIBOR at expiration times 90/360 and 1 be the face value of the contract. Then y is the value at the expiration of the futures of a long position in the underlying 90-day Eurodollar time deposit and a short position in the futures. This value is supposed to be non-stochastic and zero, which is the case for virtually all other futures contracts. The only solution, $y = 0$, implies that the interest rate (LIBOR) is zero. Moreover, there is no simple adjustment to the problem to provide a solution. For example, suppose we allow the holding of λ futures contracts, where λ is usually referred to as the *hedge ratio*. Then the problem becomes $y = 1/(1 + x) + \lambda(1 - x)$. Setting y to zero and solving gives $\lambda = -1/[(1 - x)(1 + x)]$. While this is a mathematical solution, it is not a practical solution. The value of λ is the number of futures contracts that must be traded at the start of the transaction. It cannot depend on the random outcome x.

These results do not imply that an arbitrage profit is possible. Rather, they imply than *an arbitrage transaction is not possible*. More precisely, the *standard futures arbitrage transaction* – buying the spot and selling the futures – is not a risk-free combination that would ordinarily produce a risk-free return and force the futures price to equal the spot price plus the appropriate cost of carry. For Eurodollar futures, this implies a result that contrasts with a widely-held belief: in the absence of market imperfections and the daily settlement, the rate implied by the Eurodollar futures contract is the implied forward rate in the Eurodollar term structure. That is not the case.

In this paper, I confirm a previously published result that the rate implied by the Eurodollar futures contract is not the implied forward rate in the Eurodollar term structure. This result is shown in a different and somewhat simpler format than has been previously shown. The primary contribution of this paper, however, is to examine the implications of this discrepancy between the so-called futures rate and the implied forward rate for a variety of hedging problems. I show that a number of common hedging scenarios under perfect market conditions do not provide perfect hedges, as has been typically thought. Empirical estimates are provided of the distortion caused by this unusual contract feature, and the results show a standard deviation of 17-18 basis points for contracts on 90-day LIBOR and about twice that much if the futures contract had been based on 180-day LIBOR. Since perfect hedges are this far from perfection, it can be reasonably assumed that imperfect hedges are even further away.

This paper is organized in the following manner. Section 2 provides a more formal specification of the problem. Section 3 examines the nature of arbitrage when Eurodollar futures are used, and Section 4 examines the implications for hedging with Eurodollar futures. Section 5

[1]The only other exception, which we address later, is the Chicago Board of Trade's Federal Funds futures contract.

provides empirical estimates of the magnitude of these distortions in hedges involving Eurodollar futures. Section 6 provides conclusions.

2. Background of the Problem

A. Characteristics and Background of the Eurodollar Futures Contract

The Chicago Mercantile Exchange's Eurodollar futures contract is based on the interest rate on a 90-day Eurodollar time deposit. The latter is a 90-day loan made in dollars by one bank to the other in London. While dollar loans are traded in various parts of the world, London is the center of the Eurodollar industry. These loans are characterized by bid and ask rates, called the London Interbank Bid Rate or LIBID and the London Interbank Offer Rate or LIBOR. The futures contract, and indeed nearly all Eurodollar-based derivatives, is based only on the latter rate. While London banks continuously quote these rates for standard maturities of one to twelve months, this contract is based only on the three-month or 90-day loan. At 11:00 a.m. London time (5:00 a.m. Chicago time) on the contract expiration day, the CME uses the official LIBOR as published by the British Bankers Association (BBA), which is based on a sample of quotes from 16 London banks. The BBA averages the quotes of the middle eight banks. Denote this final official LIBOR as r. The futures price at expiration is then set at 100 – r(90/360). The contract then settles at expiration with a cash transfer based on the price change from the previous day.[2] The face value of the contract is $1 million. The expiration day is the second London business day prior to the third Wednesday of the month. For many years after the creation of the contract, the only available expirations were the months of March, June, September, and December. Now, however, the CME offers Eurodollar futures contracts expiring the four nearest serial contract months as well as on the March-June-September-December cycle.[3] In recent years, expirations have gone out ten years. The margin requirements are changed from time to time but generally are less than $1,000.[4]

The quoted price of the contract is stated as 100 – r. Thus, for example, if r is 4%, then the quoted price is 100 – 4 = 96. The actual price, however, is 100 – 4(90/360) = 99. During the life of the contract, its quoted and actual prices change, but these prices should not be viewed as based directly on LIBOR at that time. For example, on May 3, 2002, the settlement price of the

[2]The final settlement is based on the price change from the previous day simply because the contract was settled the previous day. Since daily settlement in the futures market is just a termination of a contract with automatic reopening, the expiration day settlement is just a regular daily settlement albeit without a reopening of the contract.

[3]The term "serial contract month" refers to the current month and next three months, but excludes the standard contract cycle months of March, June, September, and December.

[4]The Chicago Mercantile Exchange actually does not use the more familiar term *margin requirement*, preferring the term *performance bond*.

June contract was quoted at 97.99. This price is interpreted as implying a LIBOR of 100 – 97.99 = 2.01. The actual contract price, however, is 100 - 2.01(90/360) = 99.4975. The rate of 2.01% should not be viewed as representative of current LIBOR.[5] With a face value of $1 million, the actual mark-to-market price is $1,000,000(99.4975/100) = $994,975.

Because of the manner in which the contract is constructed, an interest rate move of one basis point results in a $25 change in the price of the contract. Thus, if the rate implied by the contract price moves up to 2.02%, the quoted price becomes 100 - 2.02 = 97.98. The actual price is then 100 – 2.02(90/360) = 99.495, and the mark-to-market price is $1,000,000(99.495/100) = $994,950, a change of $25. The minimum price fluctuation is one-half tick or $12.50, except that in the delivery month the minimum price fluctuation is one-quarter tick or $6.25.

The manner described here for constructing the price of the contract, given a rate at expiration is inconsistent with the manner in which interest is calculated on the actual underlying instrument, a Eurodollar time deposit. This contract characteristic almost surely derived from the highly successful (at that time) Treasury bill futures contract. The T-bill contract was launched on January 6, 1976 and was arguably the first successful interest rate futures contract.[6] When the CME began planning its Eurodollar contract in the early 1980s, it undoubtedly drew on the success of the T-bill contract for ideas on how a contract should be designed. The T-bill contract, however, was based on a 90-day U. S. Government Treasury bill, which trades in the spot market as a discount instrument. That is, given a discount rate of r, the price of a 90-day T-bill is 100 – r(90/360). Since the T-bill contract calls for physical delivery of a T-bill at expiration, the futures contract is automatically priced at expiration at 100 – r(90/360). In other words, convergence is guaranteed, and the T-bill futures contract automatically trades as though it were based on a discount instrument.[7] In the first full year in which the Eurodollar contract traded, the T-bill contract averaged over 27,000 contracts a day, but volume began to fall as the Eurodollar contract became more widely used. In the year 2001 the T-bill contract traded an average of only about 120 contracts per day.

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[5]Current 90-day LIBOR at that time was about 2.13%. The current futures price should not imply current LIBOR any more than the current S&P 500 futures price should imply the current S&P 500 value, nor will this rate imply forward LIBOR, as is evidently believed.

[6]In 1975 the Chicago Board of Trade had created the first interest rate futures contract based on GNMA pass-throughs, but it was not successful. The contract was modified and re-introduced several times but never succeeded in generating much trading volume.

[7]Except for transaction costs, taxes, delivery options, etc. convergence is always guaranteed for contracts calling for physical delivery of the underlying. That is, if two parties engage in a contract with an infinitesimal time to expiration, the long (short) knows that he will receive (deliver) the underlying an instant later. Therefore, the transaction is equivalent to a spot transaction.

In contrast to T-bills, Eurodollar time deposits pay interest in the add-on manner. If r is the rate on 90-day LIBOR, a time deposit of \$1 million will grow to a value of \$1,000,000(1 + r(90/360)) at its maturity.

When the CME designed the Eurodollar futures contract, it faced several important decisions. One was whether to use physical delivery or cash settlement and another was whether to use the discount or add-on pricing function.

Physical delivery of a Eurodollar time deposit is impractical. These instruments do not trade in a secondary market. Hence, the only parties who could hold short positions, and therefore deliver the underlying would be London banks. This feature, if adopted, would have severely restricted participation in this market. Thus, the contract was structured to settle in cash. When cash settlement is used on a futures contract, the exchange has to designate a manner in which the final settlement price is determined. Undoubtedly relying on the success of its T-bill contract and that contract's familiarity to traders, the CME elected to use the discount method. As described above, the CME specified that the official settlement price at expiration would be 100 – r, which, as noted, converts to an actual price of 100 – r(90/360).

The use of such a formula, in light of its inconsistency with the price of the underlying, may seem strange, but looking back from the current day and age of exotic derivative instruments might well be viewed as ahead of its time. Consider for example, an ordinary call option on a stock. When the stock price at expiration is S_T, the payoff of a standard European option is $\mathrm{Max}(0,S_T - X)$ where X is the exercise price. Alternative varieties of options, however, include lookback options in which the payoff might be $\mathrm{Max}(0,S^{high} - X)$ where S^{high} is the highest price the stock achieved during the option's life, and Asian options in which the payoff is $\mathrm{Max}(0,S_{avg} - X)$ where S_{avg} is the average stock price over the life of the option. These two types of options are among the most popular of the genre of instruments referred to as exotic options. Yet, they are quite complex, and the characteristics of portfolios to price and hedge these options are complicated.

In designing the Eurodollar futures contract in this manner, the CME created, in some ways, an exotic futures contract. This payoff formula, however, allows traders to easily keep track of gains and losses. Each basis point move or price tick is always worth \$25.[8] In contrast,

[8]The Eurodollar futures contract is the first futures contract in which the futures price was structured by design to not converge to the spot price. The second, and only other contract, is the Chicago Board of Trade's Federal Funds futures contract, which is designed to pay off based on the average daily Federal Funds rate during the expiration month. Thus, the futures rate at expiration does not converge to the spot rate at expiration. Hence, it is similar in structure to an Asian style option, albeit a futures instead of an option. In the sense that it is based on a rate and not on a price, it is similar to the Eurodollar contract. This contract, however, has been nowhere near as successful as the Eurodollar contract. For every Federal

if the CME had designed the contract as an add-on instrument, its price sensitivity per basis point would not be constant.[9]

As noted, the contract has, by all measures, been highly successful, and on April 5, 1990 the CME launched a similar product based on 30-day LIBOR. Figures 2(a) and 2(b) show the annual volume of trading of these contracts over their entire lifetimes. From 1998 on, the 90-day contract traded over 100 million contracts per year or an average of over 400,000 per day. The 30-day contract traded as many as 1.8 million contracts in 1995 but has fallen below one million in some years. The 90-day contract has experienced a compound annual growth rate of 40% since its first year of 1982 and 15% since 1990. These data are based on the Commodity Futures Trading Commission's fiscal year, which ends on September 30. As of September 30, 2001, the open interest in the 90-day contract was 4,032,631 contracts with another 36,422 of the 30-day contracts open.

The 90-day Eurodollar contract's volume of about 184 million in 2001 stands in sharp contrast to its closest competitors on U. S. futures markets, the CME's own S&P 500 contract (62 million), the Chicago Board of Trade's Treasury Bond contract (58 million), the New York Mercantile Exchange's Crude Oil contract (37 million).[10] In addition, the Eurodollar contract has spawned several other successful contracts, such as options on the Eurodollar futures (volume of 69 million contracts in 2001) and a Euroyen contract (volume of about half a million contracts in 2001).[11]

Anecdotal evidence suggests that the success of the Eurodollar futures contract has largely derived from the tremendous success of what appears to be a competing market, the over-the-counter LIBOR-based derivative products market. This market consists primarily of interest rate swaps, but also includes interest rate options, typically known as caps and floors, and forward contracts, known as forward rate agreements or FRAs. In contrast to the organized and heavily regulated futures markets in which trades are recorded and reported, over-the-counter contracts

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Funds futures contract that trades, about 45 Eurodollar contracts trade. Even after considering that the Federal Funds contract is five times the size of the Eurodollar contract, the difference in activity is still quite large.

[9]To illustrate, with the contract designed as a discount instrument, its price at expiration is $1 - r(90/360)$ per \$1 face value. The first derivative with respect to r is $-90/360$, which is a constant amount and equivalent to \$25 per basis point on a \$1 million face value contract. If the contract were designed as a discount instrument, a \$1 million face value contract would be priced at $1/(1 + r(90/360))$. Then the derivative with respect to r would be $-(90/360)/(1 + r(90/360))^2)$ and would vary with the level of r.

[10]The S&P contract figures include its E-mini version, which is smaller and trades only electronically on the CME's GLOBEX2 system, but has greater volume than the pit trading version.

[11]The Eurodollar futures contract is not, however, the most active contract worldwide, at least in terms of number of contracts. The German government euro-denominated bond contract has had greater volume in

are private transactions. Hence, gauging the size of this market is difficult. The Bank for International Settlements (BIS) of Basel, Switzerland conducts semiannual surveys of global over-the-counter dealers.[12] As of June 2001 the BIS estimates that the total notional principal of all interest rate derivative products denominated in dollars is about $23.1 trillion. Including all currency denominations, swaps make up 76% of interest rate derivatives, options make up about 14%, and FRAs make up about 10%. Hence, it is probably reasonable to estimate that the notional principal of dollar-denominated interest rate swaps is about $17.6 trillion (76% of $23.1 trillion), of dollar-denominated interest rate options is about $3.2 trillion (14% of $23.1 trillion), and of dollar-denominated FRAs is about $2.3 trillion (10% of $23.1 trillion). Of course, not all dollar-based over-the-counter interest rate products are based on LIBOR, but the majority of these contracts use this rate. FRAs are the most comparable instruments to the Eurodollar futures contract. The average daily notional principal of Eurodollar futures for the year 2001 was about $4 trillion. Thus, the Eurodollar futures market is larger than the dollar-denominated FRA market but only about one-sixth the size of the overall dollar-denominated interest rate derivatives market.

Yet the Eurodollar futures market has benefited greatly from the growth of the over-the-counter market. It is widely known that dealers in interest rate over-the-counter instruments use Eurodollar futures to hedge their positions. One reason these instruments can serve well as a hedging device for derivatives dealers is that they have no convexity. With sophisticated software and term structure models, dealers can easily estimate the first-order sensitivity of a derivatives position, formally called the delta, and balance it against the aforementioned $25 move in the price of a Eurodollar futures contract per $1 move in LIBOR. The second-order sensitivity, formally called convexity or gamma, of the Eurodollar futures with respect to LIBOR is zero.[13] Hence, Eurodollar futures can be added to a derivatives position to hedge the first-order or delta effect without disturbing an already established gamma-neutral position.[14]

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recent years. This contract covers only €100,000, however and is, thus, only about 1/10 the size of the Eurodollar futures contract.

[12]These surveys are available on its web site www.bis.org.

[13]It should be noted that we are referring to LIBOR at a horizon date (forward LIBOR) and not current LIBOR. To illustrate, suppose a dealer enters into a swap to pay a fixed rate of g and receive 90-day LIBOR based on $1 million notional principal. The first settlement occurs in 30 days. The dealer hedges by shorting one Eurodollar futures contract expiring in 30 days. For every basis point move in 90-day LIBOR in 30 days, the dealer receives $25 from the swap and loses $25 on the futures. Regardless of how large the movement in LIBOR is, there is no second order effect.

[14]Of course the zero gamma of a Eurodollar futures can be a disadvantage because the futures cannot be used to hedge gamma. But since the delta hedge is the more critical and many firms prefer only to monitor but not necessarily hedge their gammas, the contract is an ideal instrument.

Eurodollar futures, however, have only specific expiration dates, which rarely line up with the needs of dealers. This introduces a form of basis risk, as well as gamma risk, which requires sophisticated modeling and software to accurately capture. Nonetheless, it is well-known in the industry that dealers use Eurodollar futures to manage this risk, and dealer trading has undoubtedly contributed to the tremendous growth in contract volume as well as the CME's willingness to extend the contract expirations out to ten years, much further out than other futures contracts go.

We now turn to a more formal statement of the pricing problem.

B. Notation

Let today be time 0. The futures contract expires at time t, and the underlying is m-day LIBOR. Let $r_i(j)$ be the rate at time i, i = 0, t, of a Eurodollar time deposit maturing in j days. Hence, $r_0(t)$ and $r_0(t+m)$ are the spot rates for t- and (t+m)-day Eurodollars and define the LIBOR term structure over the range of our interest. They imply the forward rate $g_0(t)$, which is the rate observed on day 0 for an m-day Eurodollar time deposit to begin on day t. Let $B_i(j)$ be the price on day i of a Eurodollar time deposit maturing in j days paying \$1. By market convention, these prices are defined as

$$B_0(t) \equiv \frac{1}{1 + r_0(t)\left(\dfrac{t}{360}\right)},$$

$$B_0(t+m) \equiv \frac{1}{1 + r_0(t+m)\left(\dfrac{t+m}{360}\right)}.$$

Because we shall use the adjustment m/360 often, let $\tau = m/360$. Be definition, the forward price observed at time 0 for a transaction of m days starting on day t is

$$G_0(t) \equiv \frac{B_0(t+m)}{B_0(t)}$$

and is equivalent to

$$G_0(t) \equiv \frac{1}{1 + g_0(t)\tau}$$

where $g_0(t)$ is the forward rate. Because we look only at t- and (t+m)-day maturities, it is understood that the forward price is based on an m-day instrument. We denote the price of a Eurodollar futures contract expiring at t based on m-day LIBOR as $H_0(t)$.

The forward rate at time 0 is also, by definition,

$$g_0(t) \equiv \left(\frac{1}{G_0(t)} - 1\right)\left(\frac{1}{\tau}\right).$$

The futures rate is defined as

$$h_0(t) \equiv (1 - H_0(t))\left(\frac{1}{\tau}\right).$$

It is important to note that this "futures rate" is an artificial construction. It is simply a number implied by the futures price and does not necessarily refer to an actual interest rate on a loan. In fact, this rate cannot actually be earned on any transaction, which is a major theme of this paper. The forward rate, however, is not an artificial number. It is the return available from an actual arbitrage transaction.[15]

Finally, we note that by design, the futures price at expiration is set according to the formula,

$$H_t(t) = 1 - r_t(m)\tau.$$

This is the actual futures price, the one on which the settlement is based, not the quoted futures price. Hence, the futures rate at expiration is the spot rate:

$$h_t(t) = r_t(m).$$

C. A First Look at Eurodollar Cash-and-Carry Arbitrage

A standard cash-and-carry arbitrage is a transaction in which the arbitrageur buys the underlying asset in the spot market and sells a futures contract. When expiration arrives, the arbitrageur either delivers the asset in fulfillment of the obligation on the futures contract or sells the asset in the spot market and receives a cash settlement on the futures contract. In the absence of transaction costs, these transactions are ordinarily equivalent. In a cash-and-carry arbitrage with the cash settled Eurodollar futures, only the latter transaction is possible.

Of course, futures contracts are settled daily. As proven by Cox, Ingersoll and Ross (1981), this procedure can affect the prices of futures. The daily settlement, however, is not our concern in this paper. To examine the effects of how the contract payoff is designed, let us assume that the entire payoff is received at expiration.

We now examine what happens in a cash-and-carry transaction.

[15]The forward rate is the rate that should be earned if a party goes long a Eurodollar time deposit maturing in t+m days and short a Eurodollar time deposit maturing in t days. The actual forward market instrument that has this rate is an interest rate forward contract, called a *forward rate agreement* or *FRA*, which we briefly mentioned in Section 2.A.

Time 0

Invest $B_0(t+m)$ in a Eurodollar time deposit paying \$1 in t+m days.

Sell a Eurodollar futures at $H_0(t)$ that expires at t.

Time t

The Eurodollar time deposit is worth

$$\frac{1}{1 + r_t(m)\tau}.$$

The futures price is $H_t(t) = 1 - r_t(m)\tau$. The payoff of the futures is, therefore,

$$-(H_t(t) - H_0(t)) = -(1 - r_t(m)\tau - H_0(t)) = -1 + r_t(m)\tau + H_0(t).$$

The total value of the cash-and-carry arbitrage position is

$$\frac{1}{1 + r_t(m)\tau} - 1 + r_t(m)\tau + H_0(t).$$

In this expression $H_0(t)$ is not-stochastic, but $r_t(m)$ is stochastic. Hence, in contrast to most cash-and-carry arbitrage transactions under perfect market conditions, this one is not risk-free and cannot be used to establish an initial price for the futures contract, $H_0(t)$. It is also not possible to alter the quantity of futures to turn this into a risk-free arbitrage. We cannot, for example, use an unspecified quantity of λ futures, because, as noted in Section 1, λ would be dependent on the stochastic value $r_t(m)$.

If the contract were designed to pay off as an add-on instrument, the arbitrage would indeed be risk-free. The futures price at expiration would be $1/(1 + r_t(m)\tau)$, and the futures payoff would be $-1/(1 + r_t(m)\tau) + H_0(t)$. The overall value of the position would be $1/(1 + r_t(m)\tau) - 1/(1 + r_t(m)\tau) + H_0(t) = H_0(t)$, which means that the Eurodollar time deposit purchased at time 0 would have a locked-in price of $H_0(t)$ at time t. Alternatively, if this transaction were executed in T-bills and T-bill futures or if Eurodollars were priced as discount instruments, the price of the underlying at t would be $1 - r_t(m)\tau$, the price of the futures at expiration would be $1 - r_t(m)\tau$, and value of the overall position would be $1 - r_t(m)\tau - (1 - r_t(m)\tau - H_0(t)) = H_0(t)$, which is the outcome of a perfect hedge.

D. Previous Research

The condition that the futures rate equals the forward rate is equivalent to the condition that the futures is priced using the standard cost of carry model, assuming away the daily settlement. The Chicago Mercantile Exchange's literature presents examples that assume that the futures rate equals the forward rate (Chicago Mercantile Exchange (1997)). Other references to this phenomenon are in Burghardt et al (1991, pp. 35-44), Dubofsky (1992, pp. 513-516), Arditti (1996, pp. 199-200), Hull (2002, pp. 117-118), and Siegel and Siegel (1990, pp. 209-210).

The only previous reference that questions this result is Sundaresan (1991). Using the Cox, Ingersoll, Ross (1981) framework and the assumption of no daily settlement, he shows that the futures rate does not equal the forward rate. His formula for the difference, however, contains a stochastic term. Using the Cox, Ingersoll, Ross (1985) model of the term structure, he then computes Eurodollar forward and futures prices using a numerical solution to a partial differential equation. He finds that the futures rate exceeds the forward rate by 13-17 basis points for contracts expiring in 90 days and by 30-40 basis points for contracts expiring in 180 days.[16] These results are based on a set of assumed input parameters. Using actual price data for a small sample of Eurodollar futures and forward markets, he finds that forward prices exceed futures prices by an average of 50 basis points, although there could also be other factors accounting for this difference.

3. Arbitraging the Eurodollar Futures Contract

In Section 2.C we illustrated that the standard cash-and-carry arbitrage does not hold for Eurodollar futures. The standard cash-and-carry arbitrage is an example of *state-independent pricing*. If cash-and-carry arbitrage holds, as it does for virtually all other futures contracts, then a single state or outcome can be used to represent all states or outcomes. For example, consider an asset priced at S_0 that makes no cash payments. The risk-free rate is r, which compounds annually. The price of a futures contract expiring at t is $H_0(t)$. In cash-and-carry arbitrage, the arbitrageur buys the asset for S_0, sells the futures for $H_0(t)$ and holds the position until expiration, at which time he delivers the asset and receives the futures price $H_0(t)$. Alternatively, he sells the asset for the spot price S_t and receives a cash settlement on the futures of $-(H_t(t) - H_0(t))$. Given convergence of $H_t(t)$ to S_t, this equals $-S_t + H_0(t)$. In either case, the value of the overall position is $H_0(t)$. This amount was known at time 0. Therefore, the payoff at time t, $H_0(t)$, should equal the initial outlay, $S_0(t)$, compounded at the risk-free rate, i.e., $H_0(t) = S_0(t)(1 + r)^t$ where r is the

[16] Hogan and Weintraub (1994) also discuss this point in the context of the Cox, Ingersoll, and Ross model, casually noting that the difference could be as large as 10 basis points for contracts expiring in three years, but they do not present calculations.

risk-free rate for the period 0 to t. The proof is *state independent* in that any value of S_t at t will uphold the proof as well as any other value.

The pricing of nearly all other forward and futures contracts is state-independent. The pricing of options, however, is *state-dependent*.[17] Hence, models that explicitly recognize different states of the underlying at expiration, such as the binomial and Black-Scholes models, are required to obtain option prices.

State-independent futures pricing models require only the spot price, time to expiration, the interest rate and any yield or cost associated with holding the asset. They are sometimes called cost-of-carry models, because the difference between the spot and forward or futures price is determined by the net cost of carry, i.e., the interest foregone plus storage costs net of any yield on the underlying asset. They do not require the volatility of the underlying asset. In contrast, state-dependent models require the volatility of the underlying asset. They might well be called *volatility-dependent models*.

We have already demonstrated that the Eurodollar futures contract cannot be priced using a state-independent model. We now show that it can be priced using a state-dependent model. Hence, we shall be forced to specify a volatility for the underlying.

The simplest framework is to specify a basic binomial tree structure for the one-period LIBOR. We let the current one-period LIBOR be denoted as r. The next period, the one-period LIBOR will be either r^+ or r^-. We assume that r, r^+, and r^- are rates per period and, therefore, we do not concern ourselves with the 90/360 adjustment. We continue to assume a notional principal on the futures contract of \$1. At time 0, we purchase a Eurodollar time deposit that pays \$1 at the end of period 2 for a current price of B(2). One period later, this time deposit will have a value of

$$B(2)^+ \equiv \frac{1}{1+r^+} \, or$$
$$B(2)^- \equiv \frac{1}{1+r^-}.$$

A Eurodollar futures contract expiring in one period is priced today at H. One period later its value will be

$$H^+ \equiv 1 - r^+ or$$
$$H^- \equiv 1 - r^-.$$

[17]It should, therefore, not be surprising that there is no known state-independent pricing model for the aforementioned Federal Funds futures contract, which has a payoff that is not based on the rate on a Federal Funds loan but on an average of these rates over the expiration month.

In constructing binomial models of the term structure, one can either assume martingale or risk neutral probabilities of ½ or specify the volatility in such a manner that these probabilities are produced by the model.[18] We choose to set the probabilities at ½.

At time 0 we construct a combination consisting of B(2) dollars invested in Eurodollar deposits and λ futures contracts. The value of this portfolio today is

$$V = B(2),$$

which reflects the fact that the value of a futures contract when it is first written is zero. One period later, the value of this combination will be

$$V^{+} = B(2)^{+} + \lambda(H^{+} - H) = \frac{1}{1+r^{+}} + \lambda[(1-\mathrm{r}^{+}) - \mathrm{H}] \text{ or}$$

$$V^{-} = B(2)^{-} + \lambda(H^{-} - H) = \frac{1}{1+r^{-}} + \lambda[(1-\mathrm{r}^{-}) - \mathrm{H}].$$

The structure of the problem is shown in Figure 3.

A risk-free hedge can be constructed by setting V^+ equal to V^- and solving for λ to give:

$$\lambda = \frac{\left(\frac{1}{1+r^{+}}\right) - \left(\frac{1}{1+r^{-}}\right)}{r^{+} - r^{-}}, \tag{1}$$

which can be written as

$$\lambda = \frac{B(2)^{+} - B(2)^{-}}{H^{-} - H^{+}}. \tag{2}$$

In this manner, λ appears more like the standard hedge ratio when the binomial model is applied to pricing options on simple assets such as stocks. So in general, λ is based on the ratio of the spread between the next two possible prices for the derivative and the underlying.

Since the combination of Eurodollars and futures is hedged, then $V^+ = V^-$. We may then take either V^+ or V^- and equate it to the one-period compounded value of the initial value:

$$\frac{1}{1+r^{+}} + \lambda(1 - r^{+} - H) = B(2)(1+r).$$

To solve for H, we first substitute for λ and invoke the Local Expectations Hypothesis (LEH).[19] After extensive algebra, we obtain

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[18]The latter method is characteristic of the original version of the Ho-Lee (1986) model while the former is used in the Black-Derman-Toy (1990) or Heath-Jarrow-Morton (1992) models. At this point in our analysis, it is not necessary to specify a particular term structure model.

[19]The Local Expectations Hypothesis is equivalent to the condition that a term structure is arbitrage-free. In that case, the one-period-ahead forward price, G, is equivalent to the expected spot price,

$$
\begin{aligned}
H &= 1-\left(\frac{1}{2}r^{+}+\frac{1}{2}r^{-}\right) \\
&= 1-E^{Q}(r) \\
&= \frac{1}{2}(1-r^{+})+\frac{1}{2}(1-r^{-}) = E^{Q}(H^{+-}).
\end{aligned} \tag{3}
$$

The expression in parentheses on the first line is the expected spot rate at time 1 in which expectations are taken with respect to the equivalent martingale probabilities, as denoted on the second line. The third line indicates that the futures price at time zero is the expectation of the futures price at time 1, $E^Q(H^{+-})$, in this case the expectation over the outcomes H^+ and H^- where expectations are taken using the equivalent martingale probabilities. Indeed, we have just confirmed that Eurodollar futures prices are martingales, a result established elsewhere for other futures contracts that can be priced by the simpler cash-and-carry framework.[20]

From this result, we can gain some insight into the relationship between Eurodollar forward and futures prices. With G as the forward price, we analyze the difference,

$$
G-H = G-\left(1-\frac{1}{2}(r^{+}-r^{-})\right).
$$

Again, invoking the LEH and after extensive algebraic rearrangements, we obtain

$$
G-H = \frac{\frac{1}{2}(r^{+})^{2}+\frac{1}{2}(r^{-})^{2}+\frac{1}{2}(r^{+})^{2}r^{-}+\frac{1}{2}(r^{-})^{2}r^{+}}{(1+r^{+})(1+r^{-})}. \tag{4}
$$

Since each term in the numerator is positive, the forward price will exceed the future price. Recall that if the daily settlement applied, the forward price would exceed the futures price, but in this case, there is no daily settlement. The difference between the forward price and futures price is strictly a result of the manner in which the Eurodollar futures contract is settled.

Now we wish to obtain the relationship between the forward rate and the futures rate. First, let h be the futures rate and g be the forward rate. Recall that the futures rate is one minus

---

$$
G = \left(\frac{1}{2}\right)\left(\frac{1}{1+r^{+}}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{1+r^{-}}\right),
$$

where expectations are taken with respect to the martingale or risk neutral probabilities, which here are ½. In addition, the LEH implies that the expected return on any trading strategy is the one-period rate. The forward price further ahead than one period does not equal the expected spot price. Moreover, and most importantly, the LEH does not imply that the forward rate, whether one or more periods ahead, equals the expected spot rate.

[20]Recall that for a simple asset, we showed that the forward price $G_0(t)$ equals $S_0(1 + r)^t$. It may not be apparent that $H_0(t)$ equals the expected spot price under the equivalent martingale probabilities. Recall, however, that S_0 is the expectation of S_t discounted by r if expectations are taken using equivalent

the futures price, and thus, equal to $(1/2)(r^+ - r^-)$. The forward rate can be expressed as $(1 – G)/G$. First, substitute $(1/2)(1/(1 + r^+)) + (1/2)(1/(1 + r^-))$ for the forward rate and then subtract the forward rate from the futures rate. After extensive algebraic rearrangement, we obtain

$$h - g = \frac{\sigma^2}{1 + E^Q(r^{+-})}, \tag{5}$$

where σ^2 is the variance of the spot rate at time 1 and $E^Q(r^{+-})$ is the expectation of the spot rate at time 1 in which the expectation is taken using equivalent martingale probabilities.[21]

We see that the futures rate exceeds the forward rate. For standard futures contracts with no daily settlement, futures and forward prices are equivalent and futures and forward rates are equivalent. When the daily settlement is introduced, forward prices will exceed futures prices and futures rates will exceed forward rates.[22] We see here, however, that this same result is obtained even without the daily settlement. Moreover, the difference between futures and forward rates increases with the volatility of interest rates. Finally, since the expected spot rate is directly related to the level of current interest rates, it is reasonable to conclude that this differential increases the higher is the general level of current rates.

These results are consistent with those obtained by Sundaresan. His results, however, were determined by numerically solving a partial differential equation. As noted above, he found that the difference in rates is in the range of 13 to 17 basis points for contracts expiring in 90 days. It seems apparent that the differences in forward and futures rates are not large, but may be large enough to be of concern. Of course, this result has already been somewhat known. Our greater concern and the advancement provided in this study is the effect this has on hedges.

4. Hedging with Eurodollar Futures

It is widely believed that futures contracts are commonly used to hedge against the risk of movements in the underlying. Accordingly, Eurodollar futures contracts are considered to be a good hedge against LIBOR interest rate settings on loans. Illustrations often show borrowers hedging anticipated fixed-rate loans as well as floating-rate loans in which they are already engaged. Of course, we have already stated that dealers use Eurodollar futures to hedge their positions in swaps and other interest rate derivatives. It is not clear, however, whether borrowers should, can, and actually do use Eurodollar futures for hedging purposes. As we have seen in

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martingale probabilities. That is, $S_0 = E_t^Q[S_t](1 + r)^{-t}$. This holds by definition. Substituting we obtain $G_0(t) = E_t^Q[S_t]$.

[21]In the binomial model, the variance of the rate at time 1 is

$$\sigma^2 = \frac{1}{4}\left(r^+ - r^-\right)^2.$$

previous sections, the failure of the futures price to converge to the price of the underlying Eurodollar time deposit raises questions about the effectiveness of such hedges. In this section, I present models of common hedging scenarios to determine what discrepancies exist between the outcome of a hedge using Eurodollar futures in comparison to a benchmark hedge.

It should be noted that there are many other factors that can cause hedge performance discrepancies. The primary one is that the futures contract does not expire on the day on which the interest rate is set on a loan. Secondarily, there are such factors as the cost of maintaining margin accounts and funding the cash flows arising from daily settlements. We do not concern ourselves with these factors, however, as they are unrelated to the question at hand, which is the effect of the design of the contract to pay off as a discount instrument when the underlying is an add-on instrument.

It is reasonable to expect that Eurodollar futures contracts might make effective hedging instruments for loans tied to LIBOR and perhaps even for loans not tied to but correlated with LIBOR. The CME shows such examples (Chicago Mercantile Exchange (2001)) and they are commonly found in textbooks.[23] No one, however, has looked at whether these hedges can actually provide the expected results and if not, how much of a discrepancy from the ideal would be expected. In other words, how much variation is in a hedge as a result of this manner in which the contract is designed?

In this section we examine three types of typical hedges in which a borrower attempts to lock in the rate on a loan to be taken out in the future. We assume that today, time 0, the borrower anticipates taking out a loan in t days. The loan is a zero coupon note that will be paid back m days later. There are two possible scenarios. One is that the borrower has a fixed amount of cash it needs at t. The other is that the borrower has a fixed amount available at t + m that it will to use to pay back the loan. These scenarios are illustrated in Figure 4. Although the first scenario is more likely, we will nonetheless find it instructive to examine the second as well. In fact, we will see that under ideal conditions, it is only the second and less common scenario in which a perfect hedge is possible.

[22]See Cox, Ingersoll, and Ross (1981, Propositions 6 and 9) for the proof and an explanation.

[23]See for example, Siegel and Siegel (1990, pp. 211-212), Chance (2001, pp. 459-460), Arditti (1996, pp. 198-199), Ritchken (1996, pp. 265-266, 469-470) and Dubofsky (1992, pp. 516-519). Dubofsky shows that the outcomes are not exactly identical for different values of LIBOR, and Ritchken (pp. 369-470) shows that the results are not a perfect hedge but the difference is only one basis point. Hull (2002, pp. 124-126) shows a floating-rate loan hedge using a volatility-adjusted futures position, which attempts to deal with this problem.

As noted, the loan will be a zero coupon note, but there are two ways to design such a note. One is as an add-on instrument, like a Eurodollar. The other is as a discount instrument, like a Treasury bill.

The borrower will use a futures contract sold at time 0 that expires at time t. We shall examine two possible structures for the futures contract. One is that the contract is designed to pay off as an add-on instrument. The other is with the contract paying off as a discount instrument, like the actual Eurodollar or Treasury bill futures. In the case of the futures paying off like an add-on instrument, there is no such futures contract, but it is useful to examine it since it can serve as an idealized benchmark and an indication of the results that would be expected had the CME designed the contract in this manner. Hence, it can help present the case for or against the actual chosen design of the contract.

As in prior sections, let $H_0(t)$ be the futures price today and $H_t(t)$ be the futures price at expiration. Let r_t be m-day LIBOR at expiration. We do not need LIBOR at time 0 or forward LIBOR. We continue to use τ as the factor 90/360 or whatever is appropriate given the underlying rate.

In Section A, we examine the case referred to as the standard borrowing hedge in which LIBOR is the rate on the loan. In Section B, we examine a basis hedge in which the loan is at a rate other than LIBOR. There is no futures contract based on this other rate, but the borrower, relying on a presumed correlation between the borrowing rate and LIBOR, uses the futures on LIBOR to hedge. In Section C, we examine the case in which the loan rate is LIBOR plus a spread. This is probably the most common situation, since most firms are not able to borrow at LIBOR without paying a spread.[24]

Finally, we must stress that these hedges are static. They are not adjusted through time, as in a delta-type hedge commonly associated with options. We saw in Section 3 that a state-dependent model can be used to construct a perfect hedge of a Eurodollar time deposit and a Eurodollar futures. But a state-dependent model will result in a state-dependent quantity of futures relative to the underlying. Hence, some type of dynamic adjustment would be required. Such trading is commonly conducted by dealers, but typical corporate borrowers are not well equipped to do so.

A. The Standard Borrowing Hedge: Hedging a Loan Made at LIBOR

We first consider the case where the loan is taken out by issuing an add-on note, the futures contract is an add-on note, and the borrower needs \$1 at t. This hedge is illustrated in

[24]LIBOR is roughly considered to be the rate appropriate for a AA- borrower, although this could change if the credit quality of London banks changed.

Table 1. The amount paid back at t+m is not risk-free, as evidenced by the presence of the r_t term. The interesting aspect of this result, however, is that here the loan is an add-on note *and* the futures contract is designed as an add-on instrument. In other words, even in the ideal case, a perfect hedge is not possible.

Now consider the case in which the borrower wants to repay \$1 at t+m. This type of scenario might occur, for example, if the borrower will engage in another transaction, perhaps an investment, at time t that will generate \$1 at t+m. Here we need only examine results up to t. We determine whether the total cash available at t is immune from the effects of the uncertain value of LIBOR at t. This transaction is illustrated in Table 2. We see that in this case, the hedge is indeed perfect. Of course, in practice there is no add-on futures contract, but to consider one is useful since it represents the alternative that the CME could have chosen when designing the contract.

The results for the case of borrowing to produce a fixed amount by issuing an add-on note are summarized in Panel A of Table 3. We see the payback to the add-on loan when the futures is designed as an add-on instrument in comparison to the result when it is designed as a discount instrument. Panel A of Table 4 presents the results when borrowing to repay a fixed amount, the loan is an add-on loan, and the futures is designed as an add-on instrument or as a discount instrument.

Now suppose the borrower wants to issue a discount note. These results are summarized in Panel B of Table 3 for the case of borrowing to produce \$1 at t. Note that even when the futures contract is constructed as a discount note, the hedge is still not perfect. In fact the result is a complex function of LIBOR at the hedge termination date.

The results for the case of repaying \$1 at t+m are presented in Panel B of Table 4. Note that in this case, when the add-on note is hedged with an add-on futures or the discount note is hedged with the discount futures, the hedge is perfect. In the other two cases, the effect of LIBOR is the mirror image, amount to $(r_t\tau)^2/(1 + r_t\tau)$ in one case and $-(r_t\tau)^2/(1 + r_t\tau)$ in the other.

These results show that *a perfect hedge can exist only in the case in which the borrower fixes the amount to be repaid, but not in the case when the borrower wants a fixed amount of cash at the beginning of the loan.* Having a fixed amount up front is undoubtedly the more common case encountered in practice. For that case, we should, therefore, establish a benchmark based on the assumption that the perfect hedge is not possible. For the case of issuing an add-on note, we subtract the amount repaid at t+m for the case in which the futures is an add-on note from the case in which the futures is a discount note. We shall call these differences hedging errors 1 and 2, or HE1 and HE2. These hedging errors are

HE1: Hedging error, add-on note, borrowing to produce \$1:

$$(1-H_0(t))(1+r_t\tau)-H_0(t)r_t\tau-(r_t\tau)^2-(1+(1-H_0(t))(1+r_t\tau))=-(r_t\tau)^2$$

HE2: Hedging error, discount note, borrowing to produce \$1:

$$\frac{1+(1-H_0(t))(1+r_t\tau)}{(1+r_t\tau)(1-r_t\tau)}-\frac{(1-H_0(t))+(1-r_t\tau)}{1-r_t\tau}=\frac{(r_t\tau)^2}{(1+r_t\tau)(1-r_t\tau)}$$

We are not particularly interested in the signs. We are more interested in the volatilities of these values. We can think of these volatilities as the additional variation introduced or reduced by the design of the contract. This is particularly true when a perfect hedge is possible, but even when a perfect hedge is not possible, the volatilities of the hedging errors tell us the variation attributed to contract design relative to a design in which the contract is structured to pay off like the underlying. Thus, for the case of issuing an add-on note, the Eurodollar contract design, as a discount note, introduces some variation that would not have been there if the contract were designed as an add-on note. For the case of issuing a discount note, the design of the contract as a discount instrument can be thought of as the variation reduced by contract design. These hedging errors are asymmetric. HE2 is smaller than HE1 in an absolute sense. Hence, the variation introduced for hedges of add-on notes by making the contract pay off like a discount note is greater than the variation reduced for hedges of discount notes when the futures contract pays off as a discount note. As we shall see later, however, this asymmetry is so small as to be undetectable.

For the case of borrowing to repay \$1 at t+m, the hedging errors are HE3 and HE4:

HE3: Hedging error, add-on note, repaying \$1:

$$H_0(\mathrm{t})+\frac{(r_t\tau)^2}{1+r_t\tau}-H_0(\mathrm{t})=\frac{(r_t\tau)^2}{1+r_t\tau}$$

HE4: Hedging error, discount note, repaying \$1:

$$H_0(\mathrm{t})-\frac{(r_t\tau)^2}{1+r_t\tau}-H_0(\mathrm{t})=-\left(\frac{(r_t\tau)^2}{1+r_t\tau}\right)$$

We see that these hedging errors are symmetric. The variation added for hedges of add-on notes by making the contract pay off as a discount note is exactly the same as the variation reduced for hedges of discount notes by making the contract pay off as a discount note.

Let us now examine the comparative statics of these risk measures. First we differentiate with respect to τ, the maturity of the underlying Eurodollar time deposit.

$$\frac{\partial HE1}{\partial \tau} = -2r_t^2\tau < 0$$
$$\frac{\partial HE2}{\partial \tau} = \frac{2r_t^2\tau}{(1-r_t^2\tau^2)^2} > 0$$
$$\frac{\partial HE3}{\partial \tau} = \frac{2r_t^2\tau + r_t^3\tau^2}{(1+r_t\tau)^2} > 0 \tag{6}$$
$$\frac{\partial HE4}{\partial \tau} = -\left(\frac{2r_t^2\tau + r_t^3\tau^2}{(1+r_t\tau)^2}\right) < 0$$

After taking into account the signs of HE1 – HE4, if the loan is based on LIBOR of a greater number of days, τ is larger, and the hedging error is greater. In other words, if the loan were based on 180-day LIBOR rather than 90-day LIBOR, the error would be greater. This should, in turn, lead to greater risk.[25]

Now we examine the effects of the level of LIBOR at t.

$$\frac{\partial HE1}{\partial r_t} = -2r_t\tau^2 < 0$$
$$\frac{\partial HE2}{\partial r_t} = \frac{2r_t\tau^2}{(1-r_t^2\tau^2)^2} > 0$$
$$\frac{\partial HE3}{\partial r_t} = \frac{2r_t\tau^2 + r_t^2\tau^3}{(1+r_t\tau)^2} > 0 \tag{7}$$
$$\frac{\partial HE4}{\partial r_t} = -\left(\frac{2r_t\tau^2 + r_t^2\tau^3}{(1+r_t\tau)^2}\right) < 0.$$

Again, we see that the larger is LIBOR at t, the larger is the error in an absolute sense. This means that when rates are higher, the error is greater, and hence, the variation is likely to be greater.

Thus, in general the variation associated with hedging errors relative to the benchmark case is likely to be greater the larger is LIBOR at t and the longer the term associated with the chosen LIBOR. Note, however, that the length of the hedge itself is not directly relevant, though a longer hedge period does increase the uncertainty of r_t.

B. A Basis Hedge: Hedging a Loan Made at a Rate Other Than LIBOR

Now let us examine the same types of scenarios we did in Section A, but we stipulate that the borrower is hedging a different rate. Specifically, we let the borrowing rate be designated as b_t. Hence, the note will be issued as either an add-on or a discount note at the rate b_t. The futures contract, however, will still be based on the rate r_t, and structured as either an add-on or discount instrument. We refer to this transaction as a basis hedge, since the term "basis risk" is commonly used to describe the variation in a hedge attributable to a mismatch between the underlying and the futures hedging instrument.

Table 5 presents the case for borrowing \$1 at t. We see that the payoffs are complex and nonlinear functions of the initial futures price, LIBOR and the rate b_t. In no case is a perfect hedge possible. Of course, we would not expect it to be, given that we are hedging one rate with another rate. If we establish as the benchmark the case in which an add-on note is hedged with an add-on futures and a discount note is hedged with a discount futures, we obtain the following hedging errors:

HE1: Hedging error, add-on note, borrowing to produce \$1 at t:

$$(1-H_0(t)+(1-r_t\tau))(1+b_t\tau)-\left((1-H_0(t))(1+b_t\tau)+\frac{1+b_t\tau}{1+r_t\tau}\right)$$
$$=-\left(\frac{(r_t\tau)^2}{1+r_t\tau}\right)(1+b_t\tau)$$

HE2: Hedging error, discount note, borrowing to produce \$1 at t:

$$\frac{1-H_0(t)}{1-b_t\tau}+\frac{\left(\frac{1}{1+r_t\tau}\right)}{1-b_t\tau}-\left(\frac{1-H_0(t)}{1-b_t\tau}+\frac{1-r_t\tau}{1-b_t\tau}\right)$$
$$=\frac{(r_t\tau)^2}{(1+r_t\tau)(1-b_t\tau)}$$

Table 6 presents the basis hedge results for the case of repaying \$1 at t+m. Again, none of the hedges are perfect, but we would not expect them to be, given that we are hedging one rate with an instrument on another rate. The hedging errors are

[25]The fact that the error is greater does not automatically mean it has higher variance. If the error is greater because of the addition of a constant, the variance is not greater. Technically, we cannot assert that the variance is greater, but we propose that it is likely to be so, and we empirically confirm this point later.

HE3: Hedging error, add-on note, borrowing to repay \$1 at t+m:

$$H_0(t) - 1 + r_t\tau + \frac{1}{1+b_t\tau} - \left(H_0(t) - \frac{1}{1+r_t\tau} + \frac{1}{1+b_t\tau} \right)$$
$$= \frac{(r_t\tau)^2}{1+r_t\tau}$$

HE4: Hedging error, discount note, borrowing to repay \$1 at t+m:

$$1 - b_t\tau + H_0(t) - \frac{1}{1+r_t\tau} - \left(H_0(t) + r_t\tau - b_t\tau \right)$$
$$= -\left(\frac{(r_t\tau)^2}{1+r_t\tau} \right)$$

As in the standard borrowing case, the hedging errors when borrowing to repay a fixed amount are symmetric. Thus, the variation caused by contract design for an add-on note is the same as the variation reduced by contract design for a discount note. In addition, these are the same as the corresponding hedge errors for the standard case. Thus, when borrowing to repay a fixed amount, the risk associated with contract design is unrelated to the actual loan rate.

The derivatives with respect to τ are

$$\begin{aligned}
\frac{\partial HE1}{\partial \tau} &= -\left(\left(\frac{2r_t^2\tau + r_t^3\tau^2}{(1+r_t\tau)^2} \right)(1+b_t\tau) + b_t \frac{(r_t\tau)^2}{(1+r_t\tau)} \right) < 0 \\
\frac{\partial HE2}{\partial \tau} &= \frac{r_t^2\tau + r_t^3\tau^2 + r_t^2\tau(1-b_t\tau)}{((1+r_t\tau)(1-b_t\tau))^2} > 0 \\
\frac{\partial HE3}{\partial \tau} &= \frac{r_t^3\tau^2}{(1+r_t\tau)^2} > 0 \\
\frac{\partial HE4}{\partial \tau} &= -\left(\frac{r_t^3\tau^2}{(1+r_t\tau)^2} \right) < 0
\end{aligned} \tag{8}$$

These signs are consistent with the result that the larger is τ, the larger is the absolute hedging error. This means that if the loan is tied to a longer-maturity rate, the error is greater, and hence, it is likely that the variation is greater.

The derivatives with respect to r_t are

$$\begin{aligned}
\frac{\partial HE1}{\partial r_t} &= -(1+b_t\tau)\left(\frac{2r_t\tau^2 + r_t{}^2\tau^3}{(1+r_t\tau)^2}\right) < 0 \\
\frac{\partial HE2}{\partial r_t} &= \left(\frac{1}{1-b_t\tau}\right)\frac{2r_t\tau^2 + r_t{}^2\tau^3}{(1+r_t\tau)^2} > 0 \\
\frac{\partial HE3}{\partial r_t} &= \frac{r_t{}^2\tau^3}{(1+r_t\tau)^2} > 0 \\
\frac{\partial HE4}{\partial r_t} &= -\left(\frac{r_t{}^2\tau^3}{(1+r_t\tau)^2}\right) < 0
\end{aligned} \tag{9}$$

The signs are consistent with the conclusion that the higher is r_t, the greater is the absolute hedging error and the more likely it is that the variation of the error is greater.

The derivatives with respect to b are

$$\begin{aligned}
\frac{\partial HE1}{\partial b_t} &= -\left(\frac{(r_t\tau)^2}{1+r_t\tau}\right)\tau < 0 \\
\frac{\partial HE2}{\partial b_t} &= \left(\frac{(r_t\tau)^2}{1+r_t\tau}\right)\frac{\tau}{(1-b_t\tau)^2} > 0 \\
\frac{\partial HE3}{\partial b_t} &= 0 \\
\frac{\partial HE4}{\partial b_t} &= 0
\end{aligned} \tag{10}$$

The derivatives for HE1 and HE2 imply that the higher is the loan rate, the greater the absolute hedging error, and hence, the higher the variation of the error. HE3 and HE4 are unaffected by the loan rate. Thus, when hedging a rate other than LIBOR to repay a fixed amount, the variation contributed by contract design is affected only by LIBOR and not the other rate.

C. A Spread Hedge: Hedging a Loan tied to LIBOR with a Spread

We now turn to the scenario in which the borrower takes out a loan at LIBOR plus a spread. This is probably the most common situation encountered in the corporate world. We call this the spread hedge. The spread is denoted as s and is non-stochastic. In other words, the loan is taken out at $r_t + s$. Because there is no more uncertainty in the loan rate when borrowing with a constant spread than when borrowing at LIBOR flat, we might expect it possible to construct a perfect hedge, at least for some cases.

The futures is still priced off the rate r_t. We examine the same scenarios as before. Table 7 presents the results for the spread hedge for the case of borrowing to produce \$1 at t.[26] Again, there is no perfect hedge, even for the ideal case in which the add-on futures hedges an add-on note or a discount futures hedges a discount note. Of course, we could not get a perfect hedge when borrowing at LIBOR, so this result is not surprising. The hedging errors are as follows:

HE1: Hedging error, add-on note, borrow to produce \$1 at t:

$$(1-H_0(t))+(1+r_t\tau)-H_0(t)r_t\tau-(r_t\tau)^2+s(1-H_0(t))+s(1-r_t\tau)$$
$$-\left(1+(1-H_0(t))(1+r_t\tau)+(1-H_0(t))s+\frac{s}{1+r_t\tau}\right)=-(r_t\tau)^2-s\left(\frac{(r_t\tau)^2}{1+r_t\tau}\right)$$

HE2: Hedging error, discount note, borrow to produce \$1 at t:

$$\frac{1-H_0(t)+\dfrac{1}{1+r_t\tau}}{1-r_t\tau-s\tau}-\frac{(1-H_0(t))-(1-r_t\tau)}{1-r_t\tau-s\tau}=\left(\frac{(r_t\tau)^2}{(1-r_t\tau-s\tau)(1+r_t\tau)}\right)$$

Table 8 shows the results for the case of borrowing to repay \$1 at t+m. We see that a perfect hedge is not possible when an add-on note is hedged. In the standard hedge, a perfect hedge was possible in this scenario. When a discount note is hedged with discount futures, however, the hedge is perfect, as it was in the standard case. The hedging errors are:

HE3: Hedging error, add-on note, borrowing to repay \$1 at t+m:

$$\frac{1}{1+r_t\tau+s\tau}+H_0(t)-1+r_t\tau-\left(\frac{1}{1+r_t\tau+s\tau}+H_0(t)-\frac{1}{1+r_t\tau}\right)=\frac{(r_t\tau)^2}{1+r_t\tau}$$

HE4: Hedging error, discount note, borrowing to repay \$1 at t+m:

$$1-r_t\tau-s\tau+H_0(t)-\frac{1}{1+r_t\tau}-\left(H_0(t)-s\tau\right)=-\left(\frac{(r_t\tau)^2}{1+r_t\tau}\right)$$

Interestingly, these hedging errors are unrelated to the spread and are mirror images of each other. In addition they are identical to HE3 and HE4 for the standard case as well as the basis hedge case. Thus, it does not matter if the loan is at LIBOR, LIBOR plus a spread or a totally different rate: If the hedge is designed to repay a fixed amount, rather than produce a fixed amount up front, the error resulting from contract design is unaffected by the hedge scenario.

Now we examine the derivatives with respect to τ, r_t, and s_t.

[26]It is easy to verify that when s = 0, these results reduce to those of the standard hedge.

$$\frac{\partial HE1}{\partial \tau} = -2r_t^2\tau - s\left(\frac{2r_t^2\tau + r_t^3\tau^2}{(1+r_t\tau)^2}\right) < 0$$

$$\frac{\partial HE2}{\partial \tau} = \left(\frac{r_t^2(2-s\tau)}{(1+r_t\tau)(1-r_t\tau-s\tau)^2}\right) > 0$$

$$\frac{\partial HE3}{\partial \tau} = \frac{2r_t^2\tau + r_t^3\tau^2}{(1+r_t\tau)^2} > 0 \tag{11}$$

$$\frac{\partial HE3}{\partial \tau} = -\left(\frac{2r_t^2\tau + r_t^3\tau^2}{(1+r_t\tau)^2}\right) < 0$$

These results suggest that the greater is τ, the larger is the hedging error in an absolute sense, and the variance is likely to be larger. The derivatives with respect to LIBOR are

$$\frac{\partial HE1}{\partial r_t} = -2r_t\tau^2 - s\frac{2r_t\tau^2 + r_t^2\tau^3}{(1+r_t\tau)^2} < 0$$

$$\frac{\partial HE2}{\partial r_t} = \frac{2r_t\tau^2(1-s\tau)}{((1-r_t\tau)(1-r_t\tau-s\tau))^2} > 0$$

$$\frac{\partial HE3}{\partial r_t} = \frac{2r_t\tau^2 + r_t^2\tau^3}{(1+r_t\tau)^2} > 0 \tag{12}$$

$$\frac{\partial HE4}{\partial r_t} = -\left(\frac{2r_t\tau^2 + r_t^2\tau^3}{(1+r_t\tau)^2}\right) < 0$$

Again, these results are consistent with the conclusion that the greater is r_t, the larger is the hedging error, and it is likely that the variance is larger.

The partial derivatives with respect to s are

$$\frac{\partial HE1}{\partial s} = -\left(\frac{(r_t\tau)^2}{1+r_t\tau}\right) < 0$$

$$\frac{\partial HE2}{\partial s} = \frac{(r_t\tau)^2\tau}{(1-r_t\tau)(1-r_t\tau-s\tau)^2} > 0$$

$$\frac{\partial HE3}{\partial s} = 0 \tag{13}$$

$$\frac{\partial HE4}{\partial s} = 0$$

In the first two cases, the hedging error is greater, the greater is the spread.

The hedging errors are summarized in Table 9. It is easy to see here that in the case of a basis hedge, if we change b_t to r_t, meaning that we issue the note at LIBOR, the formulas for the basis hedge turn into the formulas for the standard hedge. Also, if we let s = 0, the formulas for the spread hedge turn into the formulas for the standard hedge. We also see that the hedging errors HE3 and HE4, which apply to the case of borrowing to repay a fixed amount at t+m, indicate that it makes no difference if the hedge is a standard hedge, a basis hedge, or a spread hedge.

We now turn to an examination of the empirical magnitudes of these hedge errors that result from contract design.

5. Empirical Estimates of the Hedging Errors

The hedging errors specified in the previous section can be empirically estimated from data on spot interest rates. Data on 1-, 3-, and 6-month LIBOR are available from the Board of Governors of the Federal Reserve System at its web site www.federalreserve.gov/releases/. It contains daily observations starting on January 4, 1971. The final observation used here is on August 10, 2001. Although the Eurodollar futures contract did not begin trading until 1982, our estimates of the empirical nature of this problem are contingent on the behavior of spot LIBOR. Hence, it is acceptable to obtain data for periods of time prior to, as well as inclusive of, the period when futures contracts began trading. There are a number of days with missing data, but overall there are more than 7,800 days of data. Although the actual futures contracts in the market are 1-month and 3-month contracts, we also compute results based on the 6-month rate. This will be useful for determining the effect of a longer maturity on the underlying.

The results are calculated according to the formulas in the previous section. For HE1 and HE2, recall that the firm is borrowing to produce \$1 at t and that HE1 and HE2 are the additional interest in basis points paid at t+m resulting from the contract design not matching the structure of the note issued. Since HE1 and HE2 can be thought of as additional interest, we annualize them. For HE3 and HE4, the borrower pays back \$1 at t+m and borrows the present value of \$1 at t. The measures HE3 and HE4 are additional amounts borrowed at t. Hence, they too can be thought of as interest, albeit discount interest. Accordingly, we annualize them as well.[27]

The values of all hedging errors are of consistent signs for any value of LIBOR. We are not as interested in the mean values as in the standard deviations and the range of values. Since our interest is in the additional risk that arises from the fact that an add-on instrument is hedged with a discount instrument or vice versa, we are more interested in the volatility of the difference between the interest on these two transactions. Hence, we focus on the standard deviation. The range would also be a useful statistic, but since it is greatly influenced by extreme errors, we

[27]The values of HE1-HE4 are dollar amounts, but since the loans are for \$1 at t or t+m, it is appropriate to think of these amounts as interest, or more conveniently, as basis points.

prefer to examine the 10th and 90th percentiles and refer to these as the extrema. These measures will be considered to reflect the variation contributed by contract design.

The results for the borrowing hedge are shown in Table 10. Recall that HE1 measures the hedging error when an add-on note is issued and the futures is designed as a discount instrument. This would be analogous to the case when a Eurodollar loan is hedged with a Eurodollar futures contract. Recall that even under the best of circumstances, a perfect hedge is not possible.

For 1-month, 3-month, and 6-month LIBOR, the standard deviations are about 6, 17, and 33 basis points. The actual Eurodollar futures contract is based on 3-month LIBOR, so the most relevant result is its 17 basis point standard deviation. The extrema are 6 and 36 basis points for that instrument. Thus, a standard borrowing hedge can deviate from the idealized benchmark by a rather large amount as a result of the design of the Eurodollar futures contract. If the contract were based on 6-month LIBOR, the difference would be as high as 77 basis points with a standard deviation of 33 basis points.

HE2 measures the hedging error when a discount note is issued and the futures is designed as an add-on instrument. This is not the case for the actual Eurodollar contract, but it does serve as an interesting standard of comparison. We find, however, very similar results to the case of issuing an add-on note. The standard deviation and the 10th and 90th percentiles are approximately the same.[28] Thus, if the Eurodollar contract had been designed as an add-on contract and a discount note were issued, the variation introduced into a hedge would be about the same. Hence, hedging errors of a similar magnitude would have existed for issuers of discount notes had the CME designed the contract as an add-on instrument.

For the case of borrowing to repay a fixed amount, recall that a perfect hedge is possible. If an add-on note is issued, an add-on style futures would provide a perfect hedge. The value of HE3 gives the hedging error for an add-on note, which is the deviation of the result using discount futures from the perfect hedge benchmark. We find similar results to the case of borrowing a fixed amount. The standard deviation for 90-day LIBOR is about 16 basis points and the 10th and 90th percentiles are about 6 and 35 basis points, respectively. If a discount note is issued, a discount-style futures contract would provide a perfect hedge. As noted in Table 9, the results for HE4 are symmetric to those of HE3; hence, the variation is the same. So, had the CME designed the contract as an add-on instrument, the same variation would have been introduced for this type of hedging scenario.

[28]The formulas for HE1 and HE2 are not identical but the here the differences are less than 0.000001.

It is important to recall that the actual Eurodollar futures contract does provide a perfect hedge, but only if the hedge is designed to guarantee the amount paid back, not the amount borrowed. Hedging a fixed amount borrowed is almost surely the more common type of hedge.

To estimate the hedging errors for the case of a basis hedge, we need a different instrument to serve as the loan rate. The Federal Reserve's data base contains other rates that could be used, and from there, we choose the commercial paper and CD rates. Commercial paper is a short-term promissory note issued by a high-grade corporate borrower and sold at a discount, as in the Treasury bill market. A CD is a time deposit issued by a bank and sold as an add-on instrument, similar to a Eurodollar time deposit. When examining the results for a basis hedge, we use CDs as the instrument when we issue an add-on note and commercial paper as the instrument when we issue a discount note.

The results are shown in Table 11 and are similar to those of the standard hedge. For HE1 and HE2, the standard deviations for 1-, 3-, and 6-month rates are 6, 17, and 34 basis points. For the more realistic 6-month rate, the 10th and 90th percentiles are 5 and 37 basis points, respectively. For borrowers who need a fixed sum of money at a future date, the design of the contract reduces volatility for some borrowers as much as the alternative design would have cost other borrowers. Similar results are found for borrowers who repay a fixed sum of money. Interestingly, we conclude that even though basis risk obviously is a source of risk, its interaction with the risk associated with contract design is evidently immaterial.

In Table 12 we present the results for the case when the loan is made at a spread over LIBOR. We use spreads of 50 and 200 basis points, with the results in the top of each cell applying to the 50 basis point case and the results in the bottom of each cell applying to the 200 basis point case. HE3 and HE4 are not affected by the spread, so there is only one value to show. Again, the results are very similar to those in the previous two tables and only slightly affected by the spread. The standard deviations and 10th and 90th percentiles are extremely close to those previously reported.

Note that, as we previously proposed, in all borrowing situations, the volatility is greater the longer is the maturity and the difference is roughly proportional to the difference in maturity. Hence, the hedging error volatility for 1-month LIBOR is about a third that for 3-month LIBOR, which is about half that of 6-month LIBOR.

6. Conclusions

The highly successful Eurodollar futures contract is designed to pay off as though the underlying is a discount instrument, even though the underlying is really an add-on instrument. This feature makes it such that the futures price cannot converge to the price of the underlying.

This implies that the standard cash-and-carry transaction will not provide an arbitrage-free position that leads to the well-known cost of carry pricing model. For interest rate instruments, this also means that the futures rate, even after removing the effects of the daily settlement, will not equal the implied forward rate. The differences do, however, appear to be small, and the futures contract can be priced using state-dependent models, rather than a state-independent model such as the cost of carry model.

Hedges constructed using Eurodollar futures contracts are subject to errors that introduce uncertainty relative to a benchmark that reflects the otherwise ideal design of the futures. The benchmark is that an add-on note is hedged with an add-on futures and a discount note is hedged with a discount futures. In the former case, the benchmark represents how the Eurodollar contract might have been designed. The variation here is, therefore, a measure of the uncertainty added by contract design. In the latter case, the benchmark is how the Eurodollar contract was actually designed. The variation here is a measure of the uncertainty that would have existed had the contract been designed as an add-on instrument and can be thought of as the reduction in risk from contract design.

One interesting finding is that even under the ideally designed contract, it is not possible to obtain a perfect hedge, except when the borrower wants to pay back a fixed amount at a future date and accepts uncertainty of the amount borrowed. If the borrower wants to borrow to produce a fixed amount at a future date, there is no possible perfect hedge.

We find that when an add-on note is hedged using Eurodollar futures, the design of the contract contributes a volatility of 17-18 basis points with a potential maximum of 35-36 basis points relative to the benchmark for hedges using 90-day LIBOR. Were the futures based on 180-day LIBOR, these figures would have been about twice as large. Had the contract been designed otherwise, additional uncertainty of a similar magnitude would have manifested in hedges of discount notes.

Somewhat surprisingly, similar results are obtained for basis hedges, the more common practical scenario, in which a different rate is hedged using Eurodollar futures, and spread hedges in which the spread is 50 or 200 basis points over LIBOR. The size of the spread has little effect on our conclusions.

Because they typically have plans that dictate certain financial needs, borrowers more commonly face the scenario of borrowing to produce a fixed amount at a future date. As we show here, there is no perfect hedge in that scenario, even under ideal contract design. Given that no perfect hedge is possible, the argument against designing the contract as a discount instrument when the underlying is an add-on instrument is somewhat weaker. Thus, the CME might well have felt freer to design the contract as a discount instrument. As we see here, this design is costly for hedgers who borrow by issuing add-on notes. The alternative design would, however, be about equally costly for hedgers who borrow by issuing discount notes. It is not apparent which group of hedgers is larger.

Is the magnitude of these hedging errors sufficiently large to warrant concern? The bid-ask spread on a Eurodollar futures is 1-2 ticks, where each tick is one-half a basis point. Hence, at most the bid-ask spread amounts to one basis point, and these hedging errors are many times the magnitude of the bid-ask spread.

Corporate hedgers have shown an overwhelming preference for swaps as hedging tools. Though there are a variety of other reasons that swaps are preferred over futures, such as the attractions of customized expirations of swaps, is clear that contract design introduces some unnecessary uncertainty, at least for some hedgers. Issues of this sort are important when a futures exchange designs a new contract. Contract features are relatively permanent and introduce costs that must be weighed against benefits.

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Table 1. Standard Borrowing Hedge to Produce \$1 at t by issuing Add-on Note Hedged Using Add-on Futures

The borrower sells a futures contract at time 0, and at time t borrows an amount necessary to produce \$1 of cash at time t. The loan is an add-on note taken out at LIBOR, r_t. The futures contract is designed to pay off as an add-on instrument.

Day 0

Sell futures at $H_0(t)$

Day t

The futures price is at $H_t(t) = \frac{1}{1 + r_t\tau}$

The futures payoff is

$$= -(H_t(t) - H_0(t)) = -\left(\frac{1}{1 + r_t\tau}\right) + H_0(t) = H_0(t) - \frac{1}{1 + r_t\tau}$$

To produce \$1 in cash, the borrower takes out a loan in which

$$H_0(t) - \frac{1}{1 + r_t\tau} + B = 1,$$

where B is the amount borrowed. Then

$$B = 1 - H_0(t) + \frac{1}{1 + r_t\tau}.$$

Day t+m

The borrower pays back

$$\left(1 - H_0(t) + \frac{1}{1 + r_t\tau}\right)(1 + r_t\tau)$$
$$= 1 + (1 - H_0(t))(1 + r_t\tau).$$

Table 2. Standard Borrowing Hedge to Produce \$1 Repaid at t+m by issuing Add-on Note Hedged Using Add-on Futures

The borrower sells a futures contract at time 0, and borrows the present value of \$1 at time t, which means that it will pay back \$1 at time t+m. The loan is an add-on note taken out at LIBOR, r_t. The futures contract is designed to pay off as an add-on instrument.

Day 0

Sell futures at $H_0(t)$

Day t

The futures price is $H_t(t) = \dfrac{1}{1 + r_t \tau}$

The futures payoff is

$$-(H_t(t) - H_0(t)) = -\left(\frac{1}{1 + r_t \tau}\right) + H_0(t) = H_0(t) - \frac{1}{1 + r_t \tau}$$

In order to pay back \$1 at t+m, we borrow $\dfrac{1}{1 + r_t \tau}$

The total cash available at t is, therefore,

$$H_0(t) - \frac{1}{1 + r_t \tau} + \frac{1}{1 + r_t \tau} = H_0(t)$$

Table 3. Summary of Results for Standard Borrowing Hedge to Produce $1 at t

The value shown in the right column is the amount paid back at time t+m when the futures contract is sold at time 0, and a loan is taken out at time t in an amount such that the sum of the futures payoff and the loan proceeds equals $1. The loan is taken out as either an add-on note (A.) or a discount note (B.) at the rate r_t. The futures is constructed to pay off as either an add-on or a discount instrument at the rate r_t, as indicated in the rows.

A. Issue Add-on Note

Add-on Futures	$1 + (1 - H_0(t))(1 + r_t\tau)$
Discount Futures	$(1 - H_0(t)) + (1 + r_t\tau) - H_0(t)r_t\tau - (r_t\tau)^2$

B. Issue Discount Note

Add-on Futures	$\dfrac{1+(1-H_0(t))(1+r_t\tau)}{(1+r_t\tau)(1-r_t\tau)}$
Discount Futures	$\dfrac{(1-H_0(t))(1-r_t\tau)}{(1-r_t\tau)}$

Table 4. Summary of Results for Standard Borrowing Hedge to Produce $1 Repaid at t+m

The value shown in the right column is the amount borrowed at time t when the futures contract is sold at time 0, and a loan is taken out at time t in an amount such that the repayment of the loan at time t+m is $1. In other words, the present value of the loan at time t is the present value of $1, which is $1/(1 + r_t\tau)$. The loan is taken out as either an add-on note (A.) or a discount note (B.) at the rate r_t. The futures is constructed to pay off as either an add-on or a discount instrument at the rate r_t, as indicated in the rows.

A. Issue Add-on Note

Add-on Futures	$H_0(t)$
Discount Futures	$\dfrac{(r_t\tau)^2}{1+r_t\tau}+H_0(t)$

B. Issue Discount Note

Add-on Futures	$\dfrac{-(r_t\tau)^2}{1+r_t\tau}+H_0(t)$
Discount Futures	$H_0(t)$

Table 5. Summary of Results for Basis Hedge to Produce $1 at t

The value shown in the right column is the amount paid back at time t+m when the futures contract is sold at time 0, and a loan is taken out at time t in an amount such that the sum of the futures payoff and the loan proceeds equals $1. The loan is taken out as either an add-on note (A.) or a discount note (B.) at the rate b_t. The futures is constructed to pay off as either an add-on or a discount instrument based on the rate r_t, as indicated in the rows.

A. Issue Add-on Note

Add-on Futures	$(1 - H_0(\mathrm{t}))(1 + b_t\tau) + \dfrac{1 + b_t\tau}{1 + r_t\tau}$
Discount Futures	$(1 - H_0(t) + (1 - r_t\tau))(1 + b_t\tau)$

B. Issue Discount Note

Add-on Futures	$\dfrac{1 - H_0(t)}{1 - b_t\tau} + \dfrac{\left(\dfrac{1}{1 + r_t\tau}\right)}{1 - b_t\tau}$
Discount Futures	$\dfrac{1 - H_0(t)}{1 - b_t\tau} + \dfrac{1 - r_t\tau}{1 - b_t\tau}$

Table 6. Summary of Results for Basis Hedge to Produce \$1 Repaid at t+m

The value shown in the right column is the amount borrowed at time t when the futures contract is sold at time 0, and a loan is taken out at time t in an amount such that the repayment of the loan at time t+m is \$1. In other words, the present value of the loan at time t is the present value of \$1, which is $1/(1 + b_t\tau)$. The loan is taken out as either an add-on note (A.) or a discount note (B.) at the rate r_t. The futures is constructed to pay off as either an add-on or a discount instrument at the rate r_t, as indicated in the rows.

A. Issue Add-on Note

Add-on Futures	$H_0(t) - \frac{1}{1 + r_t\tau} + \frac{1}{1 + b_t\tau}$
Discount Futures	$H_0(t) - 1 + r_t\tau + \frac{1}{1 + b_t\tau}$

B. Issue Discount Note

Add-on Futures	$1 - b_t\tau + H_0(t) - \frac{1}{1 + r_t\tau}$
Discount Futures	$H_0(t) + r_t\tau - b_t\tau$

Table 7. Summary of Results for Spread Hedge to Produce \$1 at t

The value shown in the right column is the amount paid back at time t+m when the futures contract is sold at time 0, and a loan is taken out at time t in an amount such that the sum of the futures payoff and the loan proceeds equals \$1. The loan is taken out as either an add-on note (A.) or a discount note (B.) at the rate r_t + s. The futures is constructed to pay off as either an add-on or a discount instrument based on the rate r_t, as indicated in the rows.

A. Issue Add-on Note

Add-on Futures	$1+(1-H_0(t))(1+r_t\tau)+(1-H_0(t))s+\frac{s}{1+r_t\tau}$
Discount Futures	$(1-H_0(t))+(1+r_t\tau)-H_0(t)r_t\tau-(r_t\tau)^2+s(1-H_0(t))+s(1-r_t\tau)$

B. Issue Discount Note

Add-on Futures	$\frac{(1-H_0(t))+\left(\frac{1}{1+r_t\tau}\right)}{1-r_t\tau-s\tau}$
Discount Futures	$\frac{(1-H_0(t))+(1-r_t\tau)}{1-r_t\tau-s\tau}$

Table 8. Summary of Results for Spread Hedge to Produce \$1 Repaid at t+m

The value shown in the right column is the amount borrowed at time t when the futures contract is sold at time 0, and a loan is taken out at time t in an amount such that the repayment of the loan at time t+m is \$1. In other words, the present value of the loan at time t is the present value of \$1, which is $1/(1 + r_t\tau)$. The loan is taken out as either an add-on note (A.) or a discount note (B.) at the rate $r_t + s$. The futures is constructed to pay off as either an add-on or a discount instrument at the rate r_t, as indicated in the rows.

A. Issue Add-on Note

Add-on Futures	$\dfrac{1}{1+r_t\tau+s\tau}+H_0(t)-\dfrac{1}{1+r_t\tau}$
Discount Futures	$\dfrac{1}{1+r_t\tau+s\tau}+H_0(t)-1+r_t\tau$

B. Issue Discount Note

Add-on Futures	$1-r_t\tau-s\tau+H_0(t)-\dfrac{1}{1+r_t\tau}$
Discount Futures	$H_0(t)-s\tau$

Table 9. Summary of Hedging Errors

Hedging Error	Standard Borrowing Hedge	Basis Hedge	Spread Hedge
HE1	$-(r_t\tau)^2$	$-\left(\frac{(r_t\tau)^2}{1+r_t\tau}\right)(1+b_t\tau)$	$-(r_t\tau)^2-\frac{s(r_t\tau)^2}{(1+r_t\tau)}$
HE2	$\frac{(r_t\tau)^2}{(1+r_t\tau)(1-r_t\tau)}$	$\frac{(r_t\tau)^2}{(1+r_t\tau)(1-b_t\tau)}$	$\frac{(r_t\tau)^2}{(1-r_t\tau-s\tau)(1+r_t\tau)}$
HE3	$\frac{(r_t\tau)^2}{1+r_t\tau}$	$\frac{(r_t\tau)^2}{1+r_t\tau}$	$\frac{(r_t\tau)^2}{1+r_t\tau}$
HE4	$-\left(\frac{(r_t\tau)^2}{1+r_t\tau}\right)$	$-\left(\frac{(r_t\tau)^2}{1+r_t\tau}\right)$	$-\left(\frac{(r_t\tau)^2}{1+r_t\tau}\right)$

Table 10. Summary Statistics for Hedging Errors for Standard Hedge using Database of Daily 1-, 3-, and 6-month LIBOR

Data are obtained from the Federal Reserve web site www.federalreserve.gov/releases/ and consist of daily interest rates from August 4, 1971 through August 10, 2001.

A. 1-month LIBOR (7,818 observations)

	Borrow $1 at t		Borrow to Repay $1 at t+m	
	Issue Add-on Note	Issue Discount Note	Issue Add-on Note	Issue Discount Note
	HE1	HE2	HE3	HE4
Mean	-0.000571	0.000571	0.000566	-0.000566
Standard deviation	0.000566	0.000566	0.000558	0.000558
10^{th} percentile	-0.001139	0.000164	0.000164	-0.001128
90^{th} percentile	-0.000164	0.001139	0.001128	-0.000164

B. 3-month LIBOR (7,819 observations)

	Borrow $1 at t		Borrow to Repay $1 at t+m	
	Issue Add-on Note	Issue Discount Note	Issue Add-on Note	Issue Discount Note
	HE1	HE2	HE3	HE4
Mean	-0.001782	0.001783	0.001735	-0.001735
Standard deviation	0.001701	0.001704	0.001629	0.001629
10^{th} percentile	-0.003564	0.000564	0.000557	-0.003461
90^{th} percentile	-0.000564	0.003564	0.003461	-0.000557

C. 6-month LIBOR (7,825 observations)

	Borrow $1 at t		Borrow to Repay $1 at t+m	
	Issue Add-on Note	Issue Discount Note	Issue Add-on Note	Issue Discount Note
	HE1	HE2	HE3	HE4
Mean	-0.003664	0.003677	0.003478	-0.003478
Standard deviation	0.003305	0.003329	0.003042	0.003042
10^{th} percentile	-0.007663	0.001191	0.001162	-0.007217
90^{th} percentile	-0.001191	0.007693	0.007217	-0.001162

Table 11. Summary Statistics for Hedging Errors for Basis Hedge using Database of Daily 1-, 3-, and 6-month LIBOR

Data are obtained from the Federal Reserve web site www.federalreserve.gov/releases/ and consist of daily interest rates from August 4, 1971 through August 10, 2001. The bank CD rate is used when issuing an add-on note and the commercial paper rate is used when issuing a discount note.

A. 1-month LIBOR (7,103 – 7,818 observations for "Issue Add-on Note"; 6,544 – 7,818 observations for "Issue Discount Note")

	Borrow $1 at t		Borrow to Repay $1 at t+m	
	Issue Add-on Note	Issue Discount Note	Issue Add-on Note	Issue Discount Note
	HE1	HE2	HE3	HE4
Mean	0.000585	0.000620	0.000566	-0.000566
Standard deviation	0.000578	0.000592	0.000558	0.000558
10^{th} percentile	0.000160	0.000160	0.000164	-0.001128
90^{th} percentile	0.001161	0.001237	0.001128	-0.000164

B. 3-month LIBOR (7,107 – 7,819 observations for "Issue Add-on Note"; 7,819 observations for "Issue Discount Note")

	Borrow $1 at t		Borrow to Repay $1 at t+m	
	Issue Add-on Note	Issue Discount Note	Issue Add-on Note	Issue Discount Note
	HE1	HE2	HE3	HE4
Mean	0.001820	0.001934	0.001735	-0.001735
Standard deviation	0.001736	0.001772	0.001629	0.001629
10^{th} percentile	0.000520	0.000550	0.000557	-0.003461
90^{th} percentile	0.003674	0.004015	0.003461	-0.000557

C. 6-month LIBOR (7,107 – 7,825 observations for "Issue Add-on Note"; 6,613 – 7,825 observations for "Issue Discount Note")

	Borrow $1 at t		Borrow to Repay $1 at t+m	
	Issue Add-on Note	Issue Discount Note	Issue Add-on Note	Issue Discount Note
	HE1	HE2	HE3	HE4
Mean	0.003736	0.003955	0.003478	-0.003478
Standard deviation	0.003368	0.003415	0.003042	0.003042
10^{th} percentile	0.001105	0.001251	0.001162	-0.007217
90^{th} percentile	0.007873	0.008242	0.007217	-0.001162

Table 12. Summary Statistics for Hedging Errors for Spread Hedge using Database of Daily 1-, 3-, and 6-month LIBOR.

Data are obtained from the Federal Reserve web site www.federalreserve.gov/releases/ and consist of daily interest rates from August 4, 1971 through August 10, 2001. Top number in cell is the result with a spread of 50 basis points over LIBOR. Bottom number in cell is the result with a spread of 200 basis points over LIBOR. Values for borrowing to repay $1 at t+m (HE3 and HE4) are unaffected by the spread, so only one value is shown.

A. 1-month LIBOR (7,818 observations)

	Borrow $1 at t		Borrow to Repay $1 at t+m	
	Issue Add-on Note	Issue Discount Note	Issue Add-on Note	Issue Discount Note
	HE1	HE2	HE3	HE4
Mean	-0.000574 -0.000582	0.000571 0.000572	0.000566	-0.000566
Standard deviation	0.000569 0.000578	0.000567 0.000567	0.000558	0.000558
10th percentile	-0.001145 -0.001162	0.000164 0.000165	0.000164	-0.001128
90th percentile	-0.000165 -0.000168	0.001139 0.001141	0.001128	-0.000164

B. 3-month LIBOR (7,819 observations)

	Borrow $1 at t		Borrow to Repay $1 at t+m	
	Issue Add-on Note	Issue Discount Note	Issue Add-on Note	Issue Discount Note
	HE1	HE2	HE3	HE4
Mean	-0.000179 -0.001817	0.001785 0.001792	0.001735	-0.001735
Standard deviation	0.001709 0.001735	0.001707 0.001713	0.001629	0.001629
10th percentile	-0.003582 -0.003635	0.000565 0.000567	0.000557	-0.003461
90th percentile	-0.000567 -0.000575	0.003572 0.003586	0.003461	-0.000557

C. 6-month LIBOR (7,825 observations)

	Borrow $1 at t		Borrow to Repay $1 at t+m	
	Issue Add-on Note	Issue Discount Note	Issue Add-on Note	Issue Discount Note
	HE1	HE2	HE3	HE4
Mean	-0.003683 -0.003738	0.003686 0.003716	0.003478	-0.003478
Standard deviation	0.003321 0.003371	0.003338 0.003365	0.003042	0.003042
10th percentile	-0.007702 -0.007817	0.001194 0.001204	0.001162	-0.007217
90th percentile	-0.001197 -0.001215	0.007713 0.007776	0.007217	-0.001162

Figure 1. The Function y = 1/(1 + x) - (1 - x)

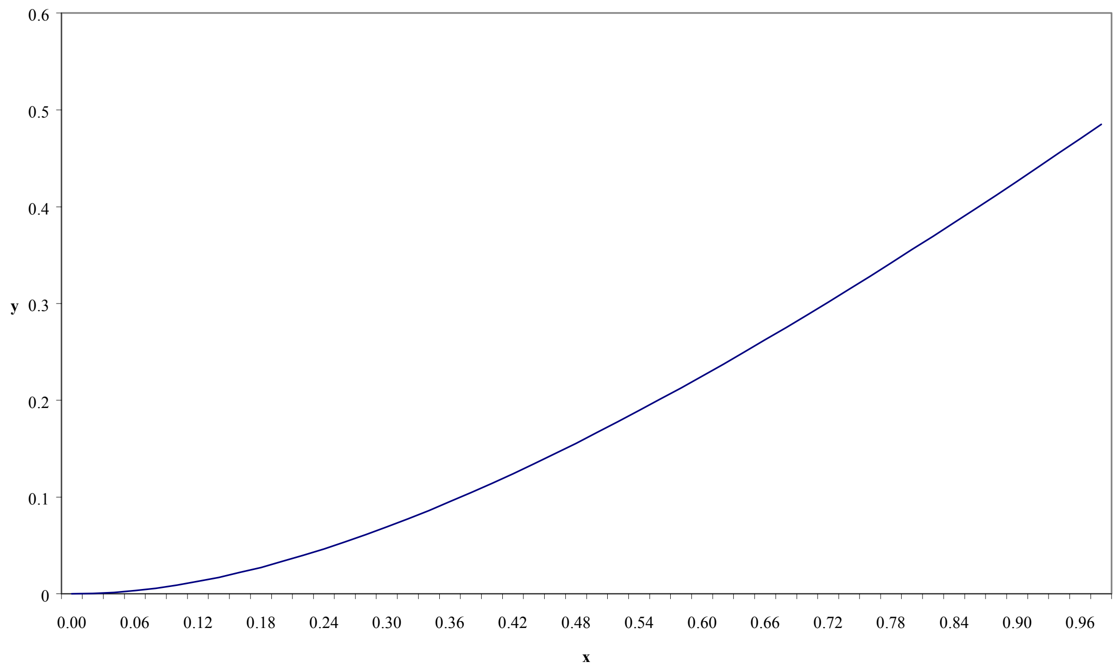
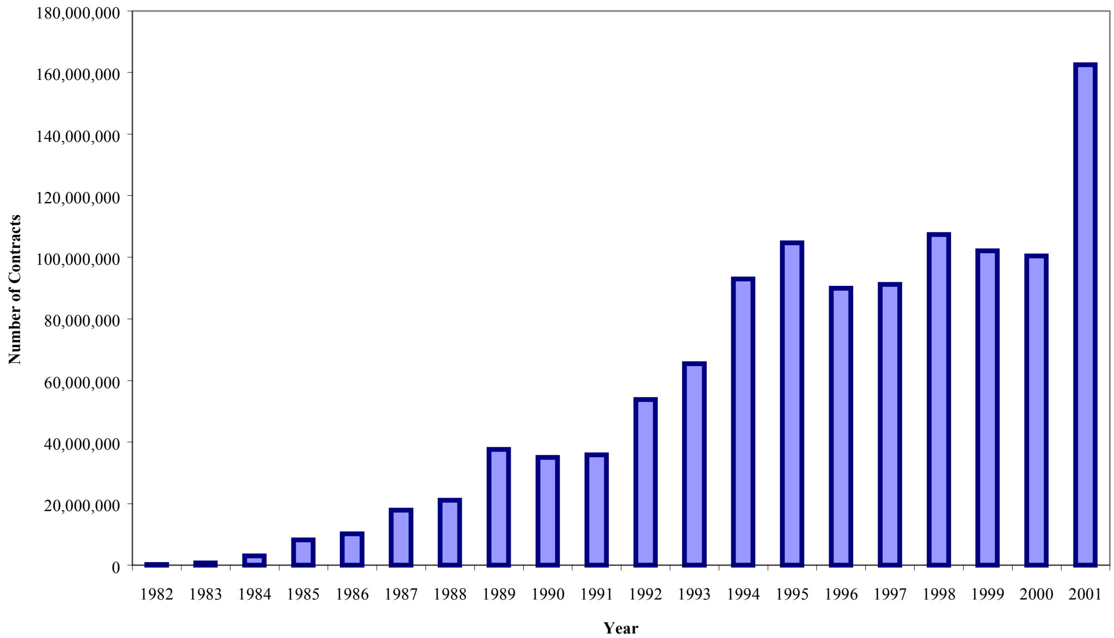
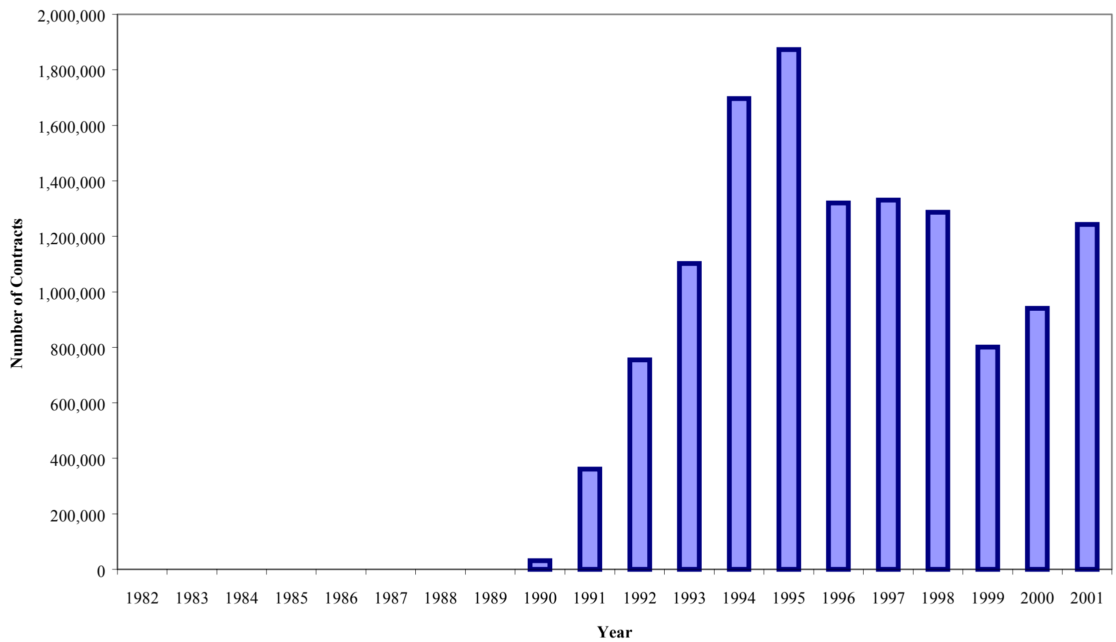


Figure 2(a): Annual Volume of Chicago Mercantile Exchange 90-day Eurodollar Futures (CFTC Fiscal Year Ending September 30)



Source: United States Commodity Futures Trading Commission, Annual Report (various years)

Figure 2(b): Annual Volume of Chicago Mercantile Exchange 30-day Eurodollar Futures
(CFTC Fiscal Year Ending September 30)



Source: United States Commodity Futures Trading Commission, Annual Report (various years)

Figure 3. Binomial Term Structure Tree for Eurodollar Time Deposit and Eurodollar Futures

$$
\begin{tikzcd}[column sep=large]
& \begin{pmatrix} r^{+} \\ B(2)^{+} = \dfrac{1}{1 + r^{+}} \\ H^{+} = 1 - r^{+} \\ V^{+} = B(2)^{+} + \lambda (H^{+} - H) \end{pmatrix} \\
\begin{pmatrix} r \\ B(2) \\ H \\ V = B(2) \end{pmatrix} \arrow[ur, dash] \arrow[dr, dash] & \\
& \begin{pmatrix} r^{-} \\ B(2)^{-} = \dfrac{1}{1 + r^{-}} \\ H^{-} = 1 - r^{-} \\ V^{-} = B(2)^{-} + \lambda (H^{-} - H) \end{pmatrix}
\end{tikzcd}
$$

Figure 4. Borrowing Scenarios